MODELLING OF MULTIPLE DELAMINATIONS IN SHELLS USING XFEM

J. Brouzoulis^{1*}, M. Fagerström¹

Department of Applied Mechanics, Chalmers University of Technology, Göteborg, Sweden *Corresponding author (jim.brouzoulis@chalmers.se)

Keywords: multiple delaminations, XFEM, shell theory

1 Introduction

There is an ever increasing interest in utilising Fiber Reinforced Polymers (FRP) in the automotive industry, especially for structural components. This calls for computational tools which can be used for the evaluation of crashworthiness. One key point is the need for computational efficiency as the models are generally very complex. Furthermore, there is a multitude of failure mechanisms which may be triggered in a layered composite, during impact or crash, with multiple delaminations being one of the primary mechanisms.

It is therefore of high importance to be able to model delaminations in a computationally efficient manner, especially for a large number of laminae. In fact, to be able to simulate the progressive failure of FRP components is a necessity for such components to be competitive within the automotive industry, as *e.g.* stated in the ERTRAC Research and Innovation Roadmap for Safe Road Transport [1].

In view of this, this work is a first step towards developing a computationally efficient shell element which can account for multiple (interlaminar) delaminations. One commonly adopted approach is to model each individual ply (or a set of plies) using several stacked shell or solid elements. However, this will lead to a large amount of degrees of freedom, especially when the number of plies increases. Another, more economical, approach is to use a single shell element which can account (internally) for the discontinuities that develop during delamination. Such an element may be constructed by enriching a suitable shell element with discontinuous shape functions in accordance with the eXtended Finite Element Method (XFEM), cf. [2] for a similar approach. Note that the present approach is similar to a layerwise model where displacements jumps are hierarchically added to the displacements field, cf. *e.g.* [3].

2 Continuous shell kinematics

To set the stage, we first briefly describe the underlying shell kinematics for a non-delaminated shell, which in the subsequent section then will be extended to allow for arbitrarily many delaminations.

2.1 Initial shell geometry and convected coordinates

As a staring point, the initial configuration B_0 of the shell is considered parameterised in terms of convected (covariant) coordinates (ξ_1, ξ_2, ξ) as

$$\mathcal{B}_0 = \left\{ \boldsymbol{X} := \boldsymbol{\Phi}(\boldsymbol{\xi}) = \bar{\boldsymbol{\Phi}}(\bar{\boldsymbol{\xi}}) + \boldsymbol{\xi} \boldsymbol{M}(\bar{\boldsymbol{\xi}}) \\ \text{with } \bar{\boldsymbol{\xi}} \in A \text{ and } \boldsymbol{\xi} \in \frac{h_0}{2}[-1,1] \right\}$$
(1)

where we introduced the contracted notation $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi)$ and $\bar{\boldsymbol{\xi}} = (\xi_1, \xi_2)$ and where the mapping $\boldsymbol{\Phi}(\boldsymbol{\xi})$ maps the inertial Cartesian frame into the undeformed configuration, cf. Figure 1. Furthermore, A is the midsurface of the inertial configuration. In Eq. (1), the mapping $\boldsymbol{\Phi}$ is defined by the midsurface placement $\bar{\boldsymbol{\Phi}}$ and the outward unit normal vector field \boldsymbol{M} (with $|\boldsymbol{M}| = 1$). The coordinate ξ is associated with this direction and h_0 is the initial thickness of the shell.

Furthermore, it should be noted that

$$d\boldsymbol{X} = (\boldsymbol{G}_{\alpha} \otimes \boldsymbol{G}^{\alpha}) \cdot d\boldsymbol{X} + \boldsymbol{M} \otimes \boldsymbol{M} \cdot d\boldsymbol{X} =$$
$$= \boldsymbol{G}_{\alpha}(\boldsymbol{\xi}) d\xi_{\alpha} + \boldsymbol{M}(\bar{\boldsymbol{\xi}}) d\xi \qquad (2)$$

whereby the covariant basis vectors are defined by

$$G_{\alpha} = \Phi_{,\alpha} + \xi M_{,\alpha} \qquad \alpha = 1,2$$
 (3)

$$G_3 = G^3 = M \tag{4}$$

where $\bullet_{,\alpha}$ denotes the derivative with respect to ξ_{α} . In addition, in Eq. (2) it was used that the contravariant basis vectors G^i are associated with the covariant vectors G_i in the normal way, *i.e.* $G_i \otimes G^i = \mathbf{I}$, leading to

$$\boldsymbol{G}_j = G_{ij}\boldsymbol{G}^i, \quad \boldsymbol{G}^j = G^{ij}\boldsymbol{G}_i$$
 (5)

with

$$G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j \text{ and } G^{ij} = (G_{ij})^{-1}$$
 (6)

Finally, the infinitesimal volume element $d\mathcal{B}_0$ of the reference configuration is formulated in the convected coordinates as

$$\mathrm{d}\mathcal{B}_0 = b_0 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi \text{ with } b_0 = (\boldsymbol{G}_1 \times \boldsymbol{G}_2) \cdot \boldsymbol{G}_3 \quad (7)$$

2.2 Current shell geometry

The current (deformed) geometry is in the present formulation described by the time dependent deformation map $\varphi(\boldsymbol{\xi}, t) \in \mathcal{B}$ of the inertial Cartesian frame as

$$\boldsymbol{x}(\boldsymbol{\xi},t) = \bar{\boldsymbol{\varphi}}(\bar{\boldsymbol{\xi}},t) + \boldsymbol{\xi}\boldsymbol{m}(\bar{\boldsymbol{\xi}},t) + \frac{1}{2}\boldsymbol{\xi}^2\gamma(\bar{\boldsymbol{\xi}},t)\boldsymbol{m}(\bar{\boldsymbol{\xi}},t)$$
(8)

where the mapping is defined by the midsurface placement $\bar{\varphi}$, the spatial director field m and an additional scalar thickness inhomogeneity strain γ , cf. Figure 1. As can be seen, the specification of the current configuration corresponds to a second order Taylor expansion along the director field, involving the inhomogeneity strain γ , thereby describing inhomogeneous thickness deformation effects of the shell. In particular, the pathological Poisson locking effect is avoided in this fashion.

To identify the corresponding deformation gradient, a relative motion dx of the placement map φ with respect to the reference placement Φ is considered as

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \text{ with } \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{X}} = \boldsymbol{g}_i \otimes \boldsymbol{G}^i \quad (9)$$

where the spatial covariant basis vectors $\boldsymbol{g}_i = \partial \boldsymbol{x} / \partial \xi_i$ are identified as

$$\boldsymbol{g}_{\alpha} = \bar{\boldsymbol{\varphi}}_{,\alpha} + \left(\xi + \frac{1}{2}\gamma\xi^2\right)\boldsymbol{m}_{,\alpha} + \frac{1}{2}\gamma_{,\alpha}\xi^2\boldsymbol{m}$$
 (10)

$$\boldsymbol{g}_3 = (1 + \gamma \xi) \, \boldsymbol{m} \tag{11}$$

3 XFEM extension for multiple delaminations

As stated above, the primary focus of the current work is to develop a shell element formulation able to represent arbitrarily many delaminations within one element. Consequently, the above basic shell kinematics need to be extended to allow for displacement and director discontinuities across each delamination interface. For this purpose, we propose herein a kinematical extension in line with the XFEM (or partition of unity concept) such that the deformation map into the spatial deformed configuration is subdivided into one continuous and one discontinuous part as

$$\boldsymbol{x}(\boldsymbol{\xi},t) = \boldsymbol{\varphi}^{\mathrm{c}}(\boldsymbol{\xi},t) + \boldsymbol{\varphi}^{\mathrm{d}}(\boldsymbol{\xi},t)$$
(12)

where the continuous part takes on the same form as the underlying non-delaminated shell element

$$\boldsymbol{\varphi}^{\mathrm{c}}(\boldsymbol{\xi},t) = \bar{\boldsymbol{\varphi}}^{\mathrm{c}}(\bar{\boldsymbol{\xi}},t) + \boldsymbol{\xi}\boldsymbol{m}^{\mathrm{c}}(\bar{\boldsymbol{\xi}},t) + \frac{1}{2}\boldsymbol{\xi}^{2}\boldsymbol{\gamma}(\bar{\boldsymbol{\xi}},t)\boldsymbol{m}^{\mathrm{c}}(\bar{\boldsymbol{\xi}},t)$$
(13)

As for the discontinuous part, it is considered as a sum of enrichments – one for each delamination – according to the XFEM, however restricted only to discontinuous enrichment of the midsurface placement and the director field. Hence, in the case of $N_{\rm del}$ delaminations through the thickness, the discontinuous part takes on the following form

$$\varphi^{d}(\boldsymbol{\xi}) = \sum_{k=1}^{N_{del}} \mathcal{H}_{S}\left(S_{k}(\boldsymbol{X},t)\right) \left(\bar{\boldsymbol{\varphi}}_{k}^{d}(\bar{\boldsymbol{\xi}},t) + \boldsymbol{\xi}\boldsymbol{m}_{k}^{d}(\bar{\boldsymbol{\xi}},t)\right)$$
$$= \mathcal{H}_{S_{k}}\left(\bar{\boldsymbol{\varphi}}_{k}^{d} + \boldsymbol{\xi}\boldsymbol{m}_{k}^{d}\right) \text{ (sum over }k)$$
(14)

In Eq. (14), $\mathcal{H}_{S}(S_{k}(\boldsymbol{X})) = \mathcal{H}_{S_{k}}$ is introduced as the standard Heaviside function pertaining to the particular delamination surface $\Gamma_{S_{k}}$. Furthermore, S_{k} is an associated level set function defining the position $\bar{\xi}_{k}$ (in thickness direction) of this surface. In particular,



Inertial configuration

Figure 1: Mappings of shell model defining undeformed and deformed shell configurations relative to inertial Cartesian frame.

 S_k is the signed distance function to the delamination interface k such that, for the current approach where we restrict the initial director field to coincide with the outward normal vector, it can be defined simply as

$$S_k = \xi - \bar{\xi}_k$$
 whereby $\frac{\partial S_k}{\partial X} = M$ (15)

where M is the normal to each delamination surface in the reference configuration. To obtain the corresponding deformation gradient, we first emphasise that

$$\frac{\partial \mathcal{H}_{S_k}}{\partial \mathbf{X}} = \frac{\partial \mathcal{H}_{S_k}}{\partial S_k} \frac{\partial S_k}{\partial \mathbf{X}} = \delta_{S_k} \mathbf{M}$$
(16)

where δ_{Sk} is the Dirac-delta function defined as

$$\int_{\mathcal{B}_0} \delta_{Sk} \bullet \mathrm{d}\mathcal{B}_0 = \int_{\Gamma_k^S} \bullet \mathrm{d}\Gamma_k^S \tag{17}$$

for any quantity •.

Consequently, in analogy with Eq. (9)-(11), the deformation gradient pertaining to the extended kinematics is obtained in the form

$$\boldsymbol{F} = \left(\boldsymbol{\varphi}^{\mathrm{c}} + \boldsymbol{\varphi}^{\mathrm{d}}\right) \otimes \boldsymbol{\nabla}_{\mathrm{X}} = \boldsymbol{F}^{\mathrm{b}} + \delta_{Sk} \boldsymbol{F}_{k}^{\mathrm{d}}$$
 (18)

where

$$\boldsymbol{F}^{\mathrm{b}} = \boldsymbol{g}_{j}^{\mathrm{b}} \otimes \boldsymbol{G}^{j}, \quad j = 1, 2, 3$$
 (19)

and

$$\boldsymbol{F}_{k}^{\mathrm{d}} = \left(\bar{\boldsymbol{\varphi}}_{k}^{\mathrm{d}} + \xi \boldsymbol{m}_{k}^{\mathrm{d}} \right) \otimes \boldsymbol{M}$$
 (20)



Figure 2: Illustration of a laminate subject to multiple delaminations.

The corresponding spatial covariant basis vectors are obtained as

$$\boldsymbol{g}_{\alpha}^{\mathrm{b}} = \bar{\boldsymbol{\varphi}}_{,\alpha}^{\mathrm{c}} + \left(\xi + \gamma \frac{1}{2}\xi^{2}\right)\boldsymbol{m}_{,\alpha}^{\mathrm{c}} + \gamma_{,\alpha}\frac{1}{2}\xi^{2}\boldsymbol{m}^{\mathrm{c}} + \sum_{k=1}^{\mathrm{N}_{\mathrm{del}}}\mathcal{H}_{S_{k}}\left(\bar{\boldsymbol{\varphi}}_{k,\alpha}^{\mathrm{d}} + \xi\boldsymbol{m}_{k,\alpha}^{\mathrm{d}}\right)$$

$$\boldsymbol{g}_{3}^{\mathrm{b}} = (1 + \gamma\xi)\boldsymbol{m}^{\mathrm{c}} + \sum_{k=1}^{\mathrm{N}_{\mathrm{del}}}\mathcal{H}_{S_{k}}\boldsymbol{m}_{k}^{\mathrm{d}}$$

$$(21)$$

4 Weak form of momentum balance

In this section, we establish the momentum balance of the shell considering the weak continuum representation of the shell applied to the shell kinematics introduced above. To arrive at the current stress resultant formulation, we start from the basic weak form of the momentum balance in terms of contributions from inertia W^{ine} , internal work W^{int} and external work W^{ext} as

Find: $\hat{\boldsymbol{n}}$ such that: $\mathcal{W}^{\text{ine}}\left(\ddot{\boldsymbol{n}};\delta\hat{\boldsymbol{n}}\right) + \mathcal{W}^{\text{int}}\left(\hat{\boldsymbol{n}};\delta\hat{\boldsymbol{n}}\right)$ (23) $+\mathcal{W}^{\text{ext}}\left(\hat{\boldsymbol{n}};\delta\hat{\boldsymbol{n}}\right) = 0 \quad \forall\delta\hat{\boldsymbol{n}}$

where we introduced the array of solution fields

$$\hat{\boldsymbol{n}}^{\mathrm{T}} = \left(\bar{\boldsymbol{\varphi}}^{\mathrm{c}}, \boldsymbol{m}^{\mathrm{c}}, \gamma, \bar{\boldsymbol{\varphi}}^{\mathrm{d}}_{k}, \boldsymbol{m}^{\mathrm{d}}_{k}\right)$$
 (24)

where, for example, $\bar{\varphi}_k^{\rm d}$ refers to the array $(\bar{\varphi}_1^{\rm d}, \bar{\varphi}_2^{\rm d}, \dots, \bar{\varphi}_{N_{\rm del}}^{\rm d})$ and represents the additional discontinuity fields associated with the existing delaminations. Furthermore, the inertia and the internal and external virtual work contributions are given as

$$\mathcal{W}^{\text{ine}} = \int_{\mathcal{B}_0} \rho_0 \left(\delta \boldsymbol{\varphi}^{\text{c}} + \delta \boldsymbol{\varphi}^{\text{d}} \right) \cdot \left(\ddot{\boldsymbol{\varphi}}^{\text{c}} + \ddot{\boldsymbol{\varphi}}^{\text{d}} \right) \mathrm{d}\mathcal{B}_0,$$
(25)

$$\mathcal{W}^{\text{int}} = \int_{\mathcal{B}_0} \left(\delta \boldsymbol{F}^{\text{T}} \cdot \boldsymbol{F} \right) : \boldsymbol{S} d\mathcal{B}_0$$
(26)

and

$$\mathcal{W}^{\text{ext}} = \int_{\mathcal{B}_0} \rho_0 \left(\delta \boldsymbol{\varphi}^{\text{c}} + \delta \boldsymbol{\varphi}^{\text{d}} \right) \cdot \boldsymbol{b} \, \mathrm{d}\mathcal{B}_0 + \int_{\partial \mathcal{B}_0} \left(\delta \boldsymbol{\varphi}^{\text{c}} + \delta \boldsymbol{\varphi}^{\text{d}} \right) \cdot \boldsymbol{t}_1 \mathrm{d}\Omega_0$$
(27)

where **b** is the body force per unit volume, $t_1 = \mathbf{P} \cdot \mathbf{N}_{\partial \mathcal{B}_0}$ is the nominal traction vector on the outer boundary $\partial \mathcal{B}_0$ and $\mathbf{P} = \mathbf{F} \cdot \mathbf{S}$ is the first Piola Kirchhoff stress tensor.

To obtain the explicit form of each individual term in Eqs. (25)-(27), we start by concluding that the inertia part is given by

$$\mathcal{W}^{\text{ine}} = \int_{\mathcal{B}_0} \rho_0 \left(\delta \boldsymbol{\varphi}^{\text{c}} + \mathcal{H}_{S_k} \delta \boldsymbol{\varphi}_k^{\text{d}} \right) \cdot \left(\ddot{\boldsymbol{\varphi}}^{\text{c}} + \mathcal{H}_{S_l} \ddot{\boldsymbol{\varphi}}_l^{\text{d}} \right) \mathrm{d}\mathcal{B}_0$$
$$= \int_{\Omega_0} \rho_0 \delta \hat{\boldsymbol{n}}^{\text{T}} (\hat{\boldsymbol{M}} \ddot{\boldsymbol{n}} + \hat{\boldsymbol{M}}_{\text{con}}) \mathrm{d}\Omega_0$$
(28)

where the consistent mass matrix \hat{M} and the convective mass force \hat{M}^{con} per unit area are obtained using a similar strategy as in Reference [4], although accounting for the alternative discontinuity enrichment. Note that, M^{con} involves contributions from the first order time derivatives of the displacement field in the inertia term of the virtual work as described in Reference [5]. Furthermore, in order to arrive at Eq. (28), a change of the integration domain from \mathcal{B}_0 (3D) to Ω_0 (2D) was made via the ratio $j_0(\xi) = b_0/\omega_0$ defining the relation between area and volumetric measures of the shell defined as

$$d\mathcal{B}_{0} = j_{0}d\xi d\Omega_{0} \text{ with} d\Omega_{0} = \omega_{0}d\xi_{1}d\xi_{2} \text{ and } \omega_{0} = |\mathbf{\Phi}_{,1} \times \mathbf{\Phi}_{,2}|$$
(29)

Furthermore, when limiting the perpendicular forces to external pressure – approximated in view of the Cauchy traction t = -pn on the deformed midsurface Ω – the external work W^{ext} can be written as

$$\mathcal{W}^{\text{ext}} = \oint_{\partial\Omega_0} \left(\delta \bar{\boldsymbol{\varphi}}^{\text{c}} \cdot \tilde{\boldsymbol{N}}^{\text{c}} + \delta \boldsymbol{m}^{\text{c}} \cdot \tilde{\boldsymbol{M}}^{\text{c}} + \delta \gamma \tilde{\boldsymbol{M}}_s^{\text{c}} \right) \mathrm{d}s$$
$$+ \oint_{\partial\Omega_0} \left(\delta \bar{\boldsymbol{\varphi}}_k^{\text{d}} \cdot \tilde{\boldsymbol{N}}_k^{\text{d}} + \delta \boldsymbol{m}_k^{\text{d}} \cdot \tilde{\boldsymbol{M}}_k^{\text{d}} \right) \mathrm{d}s$$
$$- \int_{\Omega} p \left(\delta \bar{\boldsymbol{\varphi}}^{\text{c}} + \delta \bar{\boldsymbol{\varphi}}^{\text{d}} \right) \cdot \boldsymbol{n} \, \mathrm{d}\Omega$$
(30)

where $p = p(t, \xi_1, \xi_2)$ is the external pressure, \boldsymbol{n} is the spatial normal of the deformed midsurface Ω and where $\tilde{\boldsymbol{N}}^c$, $\tilde{\boldsymbol{M}}^c$, $\tilde{\boldsymbol{M}}^c_s$, $\tilde{\boldsymbol{N}}^d_k$ and $\tilde{\boldsymbol{M}}^d_k$ are stress resultants with respect to the prescribed in-plane traction acting on the outer boundary (perpendicular to the midsurface), defined as

$$\tilde{\boldsymbol{N}}^{c} = \int_{-h_{0}/2}^{h_{0}/2} \boldsymbol{t}_{1} \mathrm{d}\boldsymbol{\xi}$$
(31)

$$\tilde{\boldsymbol{M}}^{c} = \int_{-h_{0}/2}^{h_{0}/2} \xi\left(1 + \frac{1}{2}\xi\gamma\right) \boldsymbol{t}_{1} d\xi \qquad (32)$$

$$\tilde{M}_{s}^{c} = \int_{-h_{0}/2}^{h_{0}/2} \frac{1}{2} \xi^{2} \boldsymbol{m}^{c} \cdot \boldsymbol{t}_{1} d\xi$$
 (33)

$$\tilde{\boldsymbol{N}}_{k}^{\mathrm{d}} = \int_{-h_{0}/2}^{h_{0}/2} \mathcal{H}_{S_{k}} \boldsymbol{t}_{1} \mathrm{d}\xi = \int_{\bar{\xi}_{k}}^{h_{0}/2} \boldsymbol{t}_{1} \mathrm{d}\xi$$
 (34)

$$\tilde{\boldsymbol{M}}_{k}^{\mathrm{d}} = \int_{-h_{0}/2}^{h_{0}/2} \mathcal{H}_{S_{k}} \xi \boldsymbol{t}_{1} \mathrm{d}\xi = \int_{\bar{\xi}_{k}}^{h_{0}/2} \xi \boldsymbol{t}_{1} \mathrm{d}\xi (35)$$

To obtain the explicit form of the internal virtual work contribution we first note that this contribution can be reformulated, given the present kinematical representation, as

$$\mathcal{W}^{\text{int}} = \int_{\mathcal{B}_0} \left(\delta \boldsymbol{F}^{\text{b}^{\text{T}}} \cdot \boldsymbol{F}^{\text{b}} \right) : \boldsymbol{S} \, \mathrm{d}\mathcal{B}_0 \\ + \sum_{k=1}^{\text{N}_{\text{del}}} \int_{\Gamma_k^{\text{S}}} \left(\delta \bar{\boldsymbol{\varphi}}_k^{\text{d}} + \xi \delta \boldsymbol{m}_k^{\text{d}} \right) \cdot \boldsymbol{t}_{\text{coh}}(\llbracket \bar{\boldsymbol{\varphi}}_k^{\text{d}} \rrbracket) \mathrm{d}\Omega_0$$
(36)

where $t_{\rm coh}$ is the (continuous) degrading normal traction on the delamination surface (with respect to the outward pointing normal M), which in the present approach will be represented by a cohesive zone law as a function of the delamination discontinuity

$$\llbracket \boldsymbol{\varphi}_{k}^{\mathrm{d}} \rrbracket = \left(\bar{\boldsymbol{\varphi}}_{k}^{\mathrm{d}} + \bar{\xi}_{k} \boldsymbol{m}_{k}^{\mathrm{d}} \right), \qquad (37)$$

cf. Section 5 below. Based on this, we note that the 'internal work' can be written as

$$\mathcal{W}^{\text{int}} = \int_{\Omega_0} \delta \hat{\boldsymbol{n}}_{\text{c}}^{\text{T}} \hat{\boldsymbol{N}}_{\text{c}} d\Omega_0 + \int_{\Omega_0} \delta \hat{\boldsymbol{n}}_{\text{d}}^{\text{T}} \hat{\boldsymbol{N}}_{\text{d}} d\Omega_0 + \sum_{k=1}^{N_{\text{del}}} \int_{\Gamma_k^S} \left(\delta \bar{\boldsymbol{\varphi}}_k^{\text{d}} + \xi \delta \boldsymbol{m}_k^{\text{d}} \right) \cdot \boldsymbol{t}_{\text{coh}}([\![\bar{\boldsymbol{\varphi}}_k^{\text{d}}]\!]) d\Omega_0$$
(38)

where the shell deformation and stress resultant vec-

tors have been introduced as

$$\begin{split} \delta \hat{\boldsymbol{n}}_{c}^{T} &= \left[\delta \bar{\boldsymbol{\varphi}}_{,\alpha}^{c}, \boldsymbol{m}_{,\alpha}^{c}, \delta \boldsymbol{m}^{c}, \delta \gamma_{,\alpha}, \delta \gamma \right] \\ \delta \hat{\boldsymbol{n}}_{d}^{T} &= \left[\delta \bar{\boldsymbol{\varphi}}_{k,\alpha}^{d}, \delta \boldsymbol{m}_{k,\alpha}^{d}, \delta \boldsymbol{m}_{k}^{d} \right] \\ \hat{\boldsymbol{N}}_{c}^{T} &= \left[\boldsymbol{N}^{c\,\alpha}, \boldsymbol{M}^{c\,\alpha}, \boldsymbol{T}^{c}, \boldsymbol{M}^{c\,\alpha}_{s}, T^{c}_{s} \right] \\ \hat{\boldsymbol{N}}_{d}^{T} &= \left[\boldsymbol{N}^{d\,\alpha}_{k}, \boldsymbol{M}^{d\,\alpha}_{k}, \boldsymbol{T}^{d}_{k} \right] \end{split}$$

 $(\alpha = 1, 2)$ involving the membrane, bending, shear/thickness stretch stress resultants $N^{\alpha}, M^{\alpha}, T, N_k^{d\alpha}, M_k^{d\alpha}, T_k^d$ (the three latter being conjugated with the discontinuous displacement variables) and the higher order stress resultants M_s^{α} and T_s . By introducing the abbreviation $S^{\alpha i}g_i = s_g^{\alpha}$ the explicit expressions for the stress resultants can be written

$$\boldsymbol{N}^{\alpha} = \int_{-h_0/2}^{h_0/2} \boldsymbol{s}_g^{\alpha} j_0 \mathrm{d}\boldsymbol{\xi}$$
(39)

$$\boldsymbol{M}^{\alpha} = \int_{-h_0/2}^{h_0/2} \left(1 + \frac{1}{2}\gamma\xi \right) \xi \boldsymbol{s}_g^{\alpha} j_0 \mathrm{d}\xi \tag{40}$$

$$\boldsymbol{T} = \int_{-h_0/2}^{0} \left((1+\gamma\xi) \, \boldsymbol{s}_g^3 + \frac{1}{2} \xi^2 \boldsymbol{s}_g^\alpha \gamma_{,\alpha} \right) j_0 \mathrm{d}\xi \tag{41}$$

$$M_{s}^{\alpha} = \int_{-h_{0}/2}^{h_{0}/2} \frac{1}{2} \xi^{2} \boldsymbol{s}_{g}^{\alpha} \cdot \boldsymbol{m}^{c} j_{0} \mathrm{d} \xi \qquad (42)$$
$$T = \int_{-h_{0}/2}^{h_{0}/2} (\frac{1}{2} \xi^{2} \boldsymbol{s}^{\alpha} \cdot \boldsymbol{m}^{c} + \xi \boldsymbol{s}^{3} \cdot \boldsymbol{m}^{c}) i_{0} \mathrm{d} \xi$$

$$T_s = \int_{-h_0/2} (\frac{1}{2} \xi^2 \boldsymbol{s}_g^{\alpha} \cdot \boldsymbol{m}_{,\alpha}^{c} + \xi \boldsymbol{s}_g^3 \cdot \boldsymbol{m}^{c}) j_0 \mathrm{d}\xi$$

$$\tag{43}$$

$$\boldsymbol{N}_{k}^{\mathrm{d}\,\alpha} = \int_{\bar{\xi}_{k}}^{h_{0}/2} \boldsymbol{s}_{g}^{\alpha} j_{0} \mathrm{d}\boldsymbol{\xi}$$

$$\tag{44}$$

$$\boldsymbol{M}_{k}^{\mathrm{d}\,\alpha} = \int_{\bar{\xi}_{k}}^{h_{0}/2} \boldsymbol{\xi} \boldsymbol{s}_{g}^{\alpha} j_{0} \mathrm{d}\boldsymbol{\xi}$$

$$\tag{45}$$

$$T_{k}^{d} = \int_{\bar{\xi}_{k}}^{h_{0}/2} s_{g}^{3} j_{0} d\xi$$
 (46)

Finally, by substituting the displacement field into the weak form we are given the equation of motion as

$$Ma = f^{\text{ext}} - M^{\text{con}} - b^{\text{int}} - b^{\text{coh}}$$
 (47)

where b^{int} , b^{coh} , and f^{ext} , denote internal, cohesive and external forces respectively.



Figure 3: Bilinear cohesive zone law adopted in the current paper.

5 Modelling of progressive delamination

As indicated in Section 4, the progressive interlaminar fracture process (delamination) is modelled using a cohesive zone approach. Furthermore, by introducing a cohesive zone model, interpenetration of the layers is avoided allowing for a realistic representation of the (interlaminar) kinematics.

5.1 Cohesive zone model

In this paper, a bilinear cohesive zone model, cf. Figure 3, is adopted instead of a more refined model since the focus of this study is on the (extended) kinematics. Furthermore, it keeps the modelling complexity to a minimum. Also note that the chosen cohesive law is purely elastic such that any unloading would follow the loading path in reverse; however, only examples with monotonic loading is studied in the present paper such that this non-physical unloading behaviour is avoided.

6 Numerical examples

To illustrate the proposed kinematics, three numerical examples are presented. The first example concerns simulation of the common double cantilever beam (DCB) test with the purpose of validating the kinematics of the element under progressive delamination. The second example illustrates the capability of the element formulation to kinematically represent two delaminations within a single element. The final exam-

$E_{ m L}$	126 GPa
$E_{\rm T} = E_{{\rm TT}'}$	10 GP a
$G_{\rm LT} = G_{\rm TT'}$	8 GPa
$\nu_{\rm LT} = \nu_{\rm TT'}$	0.29

Table 1: Material parameters used for the numerical examples.

ple shows that the each element can handle different number of delaminations and is thus suitable for simulation of structures undergoing substantial delaminations. In the examples, a transversely isotropic elastic material model has been used with material parameters according to Table 1. Furthermore, all laminae have a zero degree orientation.

6.1 Double cantilever beam (DCB) test

The problem consists of a beam, composed of two laminae, which has an initial crack (delamination zone) of length a = 3 mm, see Figure 4 for a description. The length of the beam is L = 200 mm, has a height of h = 3 mm and a width of w = 15 mm.

Since this example focus on the growth of a single delamination, only one set of discontinuous dofs $\{\bar{x}_{1}^{d}, m_{1}^{d}\}$ needs to be added to the solution field of the nodes within the delamination zone Γ_{k}^{S} . This will decouple the beam into its upper and lower layer. In order to model the progressive growth of the delamination a bilinear cohesive zone, as described in Subsection 5.1, is inserted between the two laminae. The fracture energy associated with mode I loading of the cohesive zone is set to $\mathcal{G}_{Ic} = 400$ N/m in this example.

The beam is modelled using 384 quadratic triangular elements with an increased mesh density in the region close to the delamination front. The free ends of the beam are subjected to prescribed displacements in the vertical direction with a magnitude p and the resulting reaction force R is registered. In Figure 5, the reaction forces corresponding to three values of the interface strength $\sigma_{\rm fn} = \{15, 30, 45\}$ MPa are shown. Also, a reference solution obtained from Euler-Bernoulli beam theory is indicated. As can be seen from the figure, the load-reaction curves correspond rather well with beam theory, more so during



Figure 4: Geometry of the DCB test used in example 1.



Figure 5: Reaction force for DCB simulation with cohesive zone.

the delamination phase. Initially, for elastic loading, the simulated curves do not follow the beam solution. The reason for this discrepancy is due to the rather low values used for $\sigma_{\rm fn}$ and $\mathcal{G}_{\rm Ic}$. However, as the strength of the cohesive zone increases, the closer the simulated curves will be to the elastic beam solution. It can thus be concluded that the proposed shell element is capable of an accurate representation of progressive delamination.

6.2 Triple cantilever beam (TCB) test

The second example illustrates the capability of the prosed element to handle two active delaminations within each element. The studied problem is a beam with the same geometry and boundary conditions as the previous example. However, in this case the beam is composed of four equal plies and where the two outermost layers have delaminated a distance of a =



Figure 6: Geometry of the TCB test used in example 2.



Figure 8: Geometry of the beam with multiple (6) delaminations with different lengths.

40 mm, see Figure 6. Furthermore, in order to put emphasis on the kinematics, no cohesive zones are active between the plies.

Beam theory gives that the reaction force necessary to vertically move the free end of one of the plies a distance of p = 1 mm is 3.1146 N. The corresponding value obtained from simulation is 3.0993 N which gives an relative error of 0.49%. In Figure 7, the mesh and displacement field near the free end of the beam is shown. Particularly notice that the layers in a given element is rendered as volume (wedge) elements stacked; nonetheless, there is only one element in the thickness direction.

6.3 Multiple delaminations

In order to show the capability of the element to handle multiple delaminations a beam with 6 delaminations, each with different lengths (a = 30 mm), is studied, cf. Figure 8. The beam is subjected to a constant edge load at the top of the beam at its free end with the magnitude 10^3 N/m in the vertical direction. As can be seen from Figure 9, multiple delamination zones of different size can be reproduced which makes the proposed element ideal for simulation structures undergoing substantial delaminations.



Figure 7: Mesh and displacement field for the triple cantilever beam test (magnification factor = 5).



Figure 9: Displacement field for beam with multiple delaminations (magnification factor = 50).

7 Concluding remarks

In this paper, the kinematics of a seven parameter shell element has been extended to handle internal delaminations. The extended element formulation, in line with the XFEM, allows for an arbitrary number of delaminations. From numerical examples, it is shown that the element is capable of accurately representing the internal discontinuities and can be used for simulation of progressive delamination. Future work includes the extension of the element to handle through the thickness cracks in addition to delaminations. Thereby, being able to capture two prominent failure mechanisms present in the failure of composites.

Acknowledgement

The research leading to these results receives funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 314182 (the MATISSE project). This publication solely reflects the authors' views. The European Community is not liable for any use that may be made of the information contained herein.

References

- [1] ERTRAC, "Research and Innovation Roadmap for Safe Road Transport," tech. rep., 2013.
- [2] R. de Borst and J. J. C. Remmers, "Computational modelling of delamination," *Composites Science* and Technology, vol. 66, pp. 713–722, May 2006.
- [3] D. H. Robbins and J. N. Reddy, "Variable kinematic modelling of laminated composite plates," *International Journal for Numerical Methods in Engineering*, vol. 39, no. November 1995, pp. 2283–2317, 1996.
- [4] S. Mostofizadeh, M. Fagerström, and R. Larsson, "Dynamic crack propagation in elastoplastic thinwalled structures: Modelling and validation," *International Journal for Numerical Methods in Engineering*, 2013.
- [5] R. Larsson, "Dynamic fracture modeling in shell structures based on XFEM," *International Journal for Numerical Methods in Engineering*, vol. 86, no. December 2010, pp. 499–527, 2011.